

# A COMBINATORIAL DESCRIPTION OF FINITE O-SEQUENCES AND ACM GENERA

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**ABSTRACT.** Our goal is to detect all the possible arithmetic genera  $g$  of a Cohen-Macaulay projective curve with a given degree  $d$ . Starting from the fact that  $g$  must belong to the range  $\{0, \dots, \binom{d-1}{2}\}$ , we develop an algorithmic procedure that avoids to construct all the possible Hilbert functions of such a curve.

We exploit essentially two results: a combinatorial description of the finite O-sequences of multiplicity  $d$  and a sort of continuity in the generation of the genera. Several experimental results show the efficiency of our method.

Moreover, we apply a search algorithm arising from our combinatorial description to single out the minimal possible Castelnuovo-Mumford regularity of a curve with Cohen-Macaulay postulation and fixed degree and genus.

## INTRODUCTION

In this paper, we face the problem of detecting all the possible arithmetic genera of Cohen-Macaulay projective curves (aCM genera, for short) of a fixed degree  $d$ . To this aim we investigate the finite O-sequences of multiplicity  $d$ . Indeed, the study of the finite O-sequences is equivalent to the study of the Hilbert functions of Cohen-Macaulay projective schemes, as a consequence of the characterization of the possible Hilbert functions of standard graded algebras given in [6] by Macaulay.

First, we provide a combinatorial description of finite O-sequences, by means of suitable connected graphs. In a very natural way, this combinatorial description provides an efficient search algorithm of the aCM genera (see Algorithm 1 in Section 3) and leads to an identification of ranges to which an aCM genus belongs, when it is produced by an O-sequence of given multiplicity and length.

More precisely, in the range  $R_d := \{0, \dots, \binom{d-1}{2}\}$ , to which the aCM genus of a Cohen-Macaulay curve of degree  $d$  must belong, we find smaller ranges  $R_d^s$ , depending not only on the multiplicity  $d$  but also on the length  $s$  of the O-sequences corresponding to the genera in  $R_d^s$ .

We determine the graph of the O-sequences of multiplicity  $d$  by using a partial order (Definition 3.6), which induces the usual order among integers on the corresponding aCM genera (Proposition 3.7 and Lemma 3.8). We extend this partial order to a total order on the O-sequences of multiplicity  $d$  and length  $s$  (Definition 3.10), which does not induce anymore the usual order among integers on the corresponding aCM genera, but which is useful to identify every range  $R_d^s$  due to the behavior of the maximal O-sequence of multiplicity  $d$  and length  $s$  (see Theorem 3.11 and Propositions 4.2 and 4.6).

Then, we single out some of the so-called *gaps*, i.e. some of the integers in  $R_d$  that are not genera of Cohen-Macaulay curves of degree  $d$ , by closed formulas (Propositions 5.3 and 5.10). Some of these gaps are located outside every range  $R_d^s$  and some other are located *near* the maximal genus in  $R_d^s$ , for values of  $s$  that are exactly determined in terms of the degree  $d$ .

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Finally, we provide an algorithm to compute all the aCM genera for a given degree  $d$ , avoiding to construct all the corresponding O-sequences (see Algorithm 2 in Section 6). The strategy supporting this algorithm combines the previous results together with a sort of *continuity* in the generation of the aCM genera that we develop in Lemma 6.1 and apply in Theorem 6.3. Note that Lemma 6.1 at least allows to detect all the genera corresponding to the non-increasing O-sequences. Due also to the above closed formulas for the gaps, a little percentage of integers of  $R_d$  remains to be checked by the search algorithm, as some experimental computations point out (see Tables 2 and 3).

In the last Section 7, we show how the search algorithm of the aCM genera can be used to detect the minimal possible Castelnuovo-Mumford regularity of a curve with Cohen-Macaulay postulation and fixed degree and genus (Proposition 7.1).

## 1. BASIC RESULTS ON HILBERT FUNCTIONS

Let  $S := K[x_0, \dots, x_n]$  be the ring of polynomials over a field  $K$  in  $n + 1$  variables and  $\mathbb{P}_K^n = \text{Proj } S$  be the  $n$ -dimensional projective space over  $K$ .

In this section we recall some basic results on the Hilbert function of a standard graded  $K$ -algebra. We refer mostly to [8].

**Definition 1.1.** Given a numerical function  $H : \mathbb{Z} \rightarrow \mathbb{Z}$  such that  $H(0) = 1$ , for each  $i \in \mathbb{N}$ :

- the  $i$ -th derivative of  $H$  is defined via:  $\Delta^0 H := H$  and, for every  $1 \leq i$ ,  $\Delta^i H(0) := 1$  and  $\Delta^i H(t) := \Delta^{i-1} H(t) - \Delta^{i-1} H(t-1)$  if  $t \geq 1$ ;
- the  $i$ -th integral of  $H$  is defined via:  $\Sigma^0 H := H$  and, for every  $i \geq 1$ ,  $\Sigma^i H(0) := 1$  and  $\Sigma^i H(t) := \Sigma^i H(t-1) + \Sigma^{i-1} H(t)$  if  $t \geq 1$ .

Given two positive integers  $a, t$ , the *binomial expansion of  $a$  in base  $t$*  is the unique writing

$$(1.1) \quad a = \binom{k(t)}{t} + \binom{k(t-1)}{t-1} + \dots + \binom{k(j)}{j}$$

where  $k(t) > k(t-1) > \dots > k(j) \geq j \geq 1$ . We use the convention for which a binomial coefficient  $\binom{n}{m}$  is null whenever  $n < m$  and  $\binom{n}{0} = 1$ , for every  $n \geq 0$ . Referring to [1, 8], we set

$$a^{(t)} := \binom{k(t)+1}{t+1} + \binom{k(t-1)+1}{t} + \dots + \binom{k(j)+1}{j+1}.$$

By an easy computation, one gets  $(a+1)^{(t)} > a^{(t)}$ .

A numerical function  $H : \mathbb{N} \rightarrow \mathbb{N}$  is *admissible* or an *O-sequence* if  $H(0) = 1$  and  $H(t+1) \leq H(t)^{(t)}$  for every  $t \geq 1$ . Thus, if  $H$  is an admissible function and  $H(t) = 0$  for some  $t$ , then  $H(t+i) = 0$  for every  $i > 0$ , and  $H$  is called a finite or Artinian O-sequence. For an Artinian O-sequence  $(h_0, \dots, h_{s-1})$  we assume  $h_{s-1} \neq 0$  and call  $s$  the *length* of the O-sequence.

**Example 1.2.** (a) Let  $H = (1, 4, 10, 18, 25, 33, 41, 50, \dots, H(t+1) = H(t) + 9, \dots)$ , we have  $\Delta^1 H = (1, 3, 6, 8, 7, 8, 8, 9, 9, \dots)$  and  $\Delta^2 H = (1, 2, 3, 2, -1, 1, 0, 1)$ , which is not admissible.

(b) Let  $H = (1, 5, 14, 29, \dots, H(t+1) = H(t) + 15, \dots)$ . In this case, we have  $\Delta^1 H = (1, 4, 9, 15, 15, \dots)$  and  $\Delta^2 H = (1, 3, 5, 6)$  is admissible.

Let  $I$  be a homogeneous ideal in  $S$ . We denote by  $I_t$  the  $K$ -vector space of the homogeneous polynomials of  $I$  of degree  $t$ .

**Definition 1.3.** The *Hilbert function* of the standard graded  $K$ -algebra  $S/I$  is the numerical function defined by  $H_{S/I} : t \in \mathbb{N} \rightarrow \dim_K S_t/I_t$ . If the Krull dimension of  $S/I$  is  $k + 1$ , the  $k$ -degree polynomial  $p_{S/I}(z) \in \mathbb{Q}[z]$  such that  $p_{S/I}(t) = H_{S/I}(t)$ ,  $t \gg 0$ , is its

*Hilbert polynomial.* The integer  $\rho_{S/I} := \min\{t \in \mathbb{N} \mid H_{S/I}(t) = p_{S/I}(t), \forall t' \geq t\}$  is the *regularity of the Hilbert function of  $S/I$* .

The *Hilbert series*  $\sum_{t \in \mathbb{N}} H_{S/I}(t)z^t$  of  $S/I$  is equal to a rational function  $\frac{h(z)}{(1-z)^{k+1}}$ , where  $h(z) = h_0 + h_1z + \cdots + h_{s-1}z^{s-1} \in \mathbb{Z}[z]$  is the  *$h$ -polynomial* of  $S/I$  and  $(h_0, h_1, \dots, h_{s-1})$  is the  *$h$ -vector* of  $S/I$ , with  $h_{s-1} \neq 0$ .

**Remark 1.4.** If  $(h_0, h_1, \dots, h_{s-1})$  is the  $h$ -vector of  $S/I$ , then  $h_0 = 1$ ,  $(h_0, h_1, \dots, h_{s-1}) = (\Delta^{k+1}H_{S/I}(0), \dots, \Delta^{k+1}H_{S/I}(s-1))$  and  $s = \rho_{S/I} + k + 1$ ; moreover, we have  $H_{S/I} = \Sigma^{k+1}(h_0, h_1, \dots, h_{s-1})$ .

A very famous result of Macaulay [6] states that, for each admissible function  $H$ , it is possible to construct a suitable lexicographic ideal  $L \subset S = K[x_0, \dots, x_n]$  such that  $H$  is the Hilbert function of  $S/L$ . In particular, a numerical function is the Hilbert function of a (standard and finite) graded  $K$ -algebra if and only if it is admissible.

**Definition 1.5.** For a standard graded  $K$ -algebra  $A$  with  $h$ -vector  $(h_0, h_1, \dots, h_{s-1})$ , the *multiplicity*  $e(A)$  both of  $A$  and of its Hilbert function  $H_A$  is defined by  $e(A) := \sum_i h_i$ .

**Remark 1.6.** Note that the multiplicity of a standard graded  $K$ -algebra is equal to the multiplicity of its  $h$ -vector. Moreover, if  $p_A(z) = \frac{d}{k!}z^k + \cdots$  is the Hilbert polynomial of  $A$ , then  $e(A) = d$  by Remark 1.4.

We end this section recalling the following relevant computational result.

**Theorem 1.7** ([5, Theorem 5.4.15 a)). *The Hilbert polynomial of a scheme of dimension  $k$  with  $h$ -vector  $\mathbf{h} = (h_0, h_1, \dots, h_{s-1})$  is*

$$(1.2) \quad p(z) = h_0 \binom{z+k}{k} + h_1 \binom{z+k-1}{k} + \cdots + h_{s-1} \binom{z+k-(s-1)}{k}.$$

## 2. HILBERT FUNCTIONS OF ARITHMETICALLY COHEN-MACAULAY CURVES

A graded  $K$ -algebra  $S/I$  with Krull-dimension  $k+1$  is a Cohen-Macaulay  $S$ -module if there is an  $S$ -regular sequence  $f_0, \dots, f_k$  of homogeneous elements in  $S/I$  (see [1] for a treatment on Cohen-Macaulay rings).

A projective subscheme  $X \subset \mathbb{P}^n$ , with (saturated) defining ideal  $I$ , is arithmetically Cohen-Macaulay (aCM, for short) if the  $K$ -algebra  $S/I$  is Cohen-Macaulay. We denote by  $H_X$  and  $p_X(z)$  the Hilbert function and the Hilbert polynomial of  $X$ , respectively, and by  $\rho_X$  the regularity of  $H_X$ . The *degree*  $\deg(X)$  of  $X$  is the multiplicity of  $S/I$ .

If  $X$  is a curve of degree  $d$ , then its Hilbert polynomial is  $dz + 1 - g$ , where  $g$  is the *arithmetic genus* of the curve.

**Definition 2.1.** A closed subscheme  $X \subset \mathbb{P}_K^n$  has *Cohen-Macaulay postulation* if there is an aCM closed subscheme  $W \subset \mathbb{P}_K^n$  such that  $H_X = H_W$ . In this case, we say that  $H_X$  is an *aCM function* and the corresponding Hilbert polynomial is an *aCM polynomial*. If moreover  $X$  is a curve, we say that its arithmetic genus is an *aCM genus*.

An aCM function is characterized by the behavior of its  $h$ -vector in the following way.

**Theorem 2.2** ([8, Theorem 1.5]). *A finite sequence  $(h_0, h_1, \dots, h_{s-1})$  of (positive) integers is the  $h$ -vector of a Cohen-Macaulay (standard and finite) graded  $K$ -algebra if and only if it is admissible.*

**Remark 2.3.** By Remark 1.4 and Theorem 2.2, it is evident that all the possible Hilbert functions of a Cohen-Macaulay graded  $K$ -algebra of multiplicity  $d$  and Krull dimension  $k+1$  can be computed from the Artinian O-sequences of multiplicity  $d$  performing the  $(k+1)$ -th integrals of the O-sequences.

**Lemma 2.4.** *The arithmetic genus of an aCM curve is non-negative.*

*Proof.* It is enough to apply Theorem 1.7. Anyway, we provide also the following elementary argument.

Let  $\mathbf{h} = (h_0, h_1, \dots, h_{s-1})$  be the  $h$ -vector of an aCM curve of degree  $d$ , i.e. a finite O-sequence (of length  $s$  and) of multiplicity  $d = \sum_i h_i$ . Then, the O-sequences  $\Sigma \mathbf{h}$  and  $\Sigma^2 \mathbf{h}$  satisfy, respectively:

$$\Sigma \mathbf{h}(i) = \begin{cases} \sum_{j=0}^i h_j, & 0 \leq i \leq s-2 \\ d, & i \geq s-1 \end{cases} \quad \Sigma^2 \mathbf{h}(i) = \begin{cases} \sum_{j=0}^i (i-j+1)h_j, & 0 \leq i \leq s-3 \\ \sum_{j=0}^{s-2} (s-1-j)h_j + d(i-s+2), & i \geq s-2 \end{cases}$$

and  $\Sigma^2 \mathbf{h}$  is the Hilbert function of a curve. The function  $\Sigma^2 \mathbf{h}$  has regularity  $s-2$  and Hilbert polynomial  $p(z) = dz + 1 - g$ , where  $g$  is the arithmetic genus of the curve. Hence, by Remark 1.4 and being  $h_0 = 1$ , we have

$$(2.1) \quad g = 1 + (s-2)d - p(s-2) = \sum_{j=2}^{s-1} (j-1)h_j \geq 0. \quad \square$$

**Lemma 2.5.** (i) *Every positive integer  $g$  is the genus of some aCM curve.*

(ii) *If  $g$  is the arithmetic genus of an aCM curve  $C_d$  of degree  $d$ , then there is also an aCM curve  $C_{d+1}$  of degree  $d+1$  with the same arithmetic genus  $g$ .*

*Proof.* (i) It is enough to take any O-sequence  $(1, h_1, g)$ , with  $h_1^{(1)} \geq g$ .

(ii) If  $\mathbf{h}_d = (1, h_1, h_2, \dots, h_{s-1})$  is the  $h$ -vector of  $C_d$ , then the sequence  $\mathbf{h}_{d+1} = (1, h_1 + 1, h_2, \dots, h_{s-1})$  is also an O-sequence and is the  $h$ -vector of a curve  $C_{d+1}$  with Hilbert polynomial  $(d+1)z + 1 - g$ . Indeed, the multiplicity of the O-sequence  $\mathbf{h}_{d+1}$  is  $d+1$  and then we apply formula (2.1), in which the integer  $h_1$  does not occur. From a geometric point of view, this means that  $C_{d+1}$  can be obtained as the union of  $C_d$  and a line through a point of  $C_d$ .  $\square$

### 3. A COMBINATORIAL DESCRIPTION OF FINITE O-SEQUENCES

In this section, we consider a natural structure on the set of all finite O-sequences, which will be useful for the algorithm procedures we will study and which will suggest some first relevant information about the aCM genera.

We let  $\mathbf{e}_i$  denote any O-sequence, of any length, consisting entirely of 0 except 1 in the  $i$ -th position. Moreover, we introduce the following notation that gives a compact way to denote some particular O-sequences:

$$(1^{\alpha_0}, h_{i_1}^{\alpha_1}, h_{i_2}^{\alpha_2}, \dots, h_{i_k}^{\alpha_k}) := (\underbrace{1, \dots, 1}_{\alpha_0 \text{ times}}, \underbrace{h_{i_1}, \dots, h_{i_1}}_{\alpha_1 \text{ times}}, \dots, \underbrace{h_{i_k}, \dots, h_{i_k}}_{\alpha_k \text{ times}})$$

**Definition 3.1.** The *O-sequences graph* is the directed graph  $\mathcal{G}$  with the following characteristics:

- the vertices  $V(\mathcal{G})$  of the graph are the finite O-sequences;
- there is an edge going from  $\mathbf{h}$  to  $\mathbf{h}'$  if  $\mathbf{h}' - \mathbf{h} = \mathbf{e}_i$  for some  $i$ , i.e.  $\mathbf{h}'$  can be obtained from  $\mathbf{h}$  by increasing  $h_i$  by one.

We label an edge of  $\mathcal{G}$  by  $\mathbf{e}_i$ , if the edge goes from  $\mathbf{h}$  to  $\mathbf{h}'$  increasing by 1 the  $i$ -th entry of the O-sequence  $\mathbf{h}$ .

Let us consider the map  $\mathcal{G} \rightarrow \mathbb{N}$  that associates to each O-sequence the genus of an aCM curve having this O-sequence as  $h$ -vector. By abuse of notation, we denote this map by  $g$ .

**Lemma 3.2.** (i) The O-sequences graph  $\mathcal{G}$  is a rooted connected graph without loops. The root is the O-sequence of multiplicity 1. Denoted by  $d_{\mathcal{G}}(\mathbf{h})$  the distance of the node  $\mathbf{h}$  from the root, we have  $d_{\mathcal{G}}(\mathbf{h}) = e(\mathbf{h}) - 1$ .  
(ii)  $g(\mathbf{h} + \mathbf{e}_i) = g(\mathbf{h}) + i - 1$ .

*Proof.* (i) For any  $\mathbf{h} = (1, h_1, \dots, h_{s-1})$ , the sequence  $\mathbf{h}' = \mathbf{h} - \mathbf{e}_{s-1}$  is admissible so that there is an edge going from  $\mathbf{h}'$  to  $\mathbf{h}$ . Repeating this procedure, we get the O-sequence (1) that cannot be the head of any edge, proving that  $\mathcal{G}$  is connected. There are no loops as each edge increases the multiplicity by 1.

(ii) Straightforward from formula (2.1).  $\square$

We would like to be able to define a subgraph  $\mathcal{T} \subset \mathcal{G}$  that turns out to be in fact a spanning tree. In this way, we can design ad hoc algorithms to visit the tree in order to quickly find the O-sequences with the properties we will look for. The idea for determining  $\mathcal{T}$  is the one used in the proof of the connectedness of  $\mathcal{G}$  in Lemma 3.2. For each node of  $\mathcal{G}$ , we consider only the edge coming from the O-sequence obtained lowering the value with the greatest index. Indeed, notice that each O-sequence  $\mathbf{h}$  (of a length  $s$ ) has a successor in  $\mathcal{T}$ , as  $\mathbf{h} + \mathbf{e}_s$  is always a finite O-sequence, whereas the sequence  $\mathbf{h} + \mathbf{e}_{s-1}$  might not be admissible.

**Definition 3.3.** We call *O-sequences tree* the subgraph  $\mathcal{T} \subset \mathcal{G}$  such that:

- $V(\mathcal{T}) = V(\mathcal{G})$ ;
- $E(\mathcal{T}) = \{(\mathbf{h}, \mathbf{h}') \in E(\mathcal{G}) \mid \mathbf{h}' = \mathbf{h} + \mathbf{e}_s \text{ or } \mathbf{h}' = \mathbf{h} + \mathbf{e}_{s-1}, \text{ if } h_{s-2}^{(s-2)} > h_{s-1}\}$ .

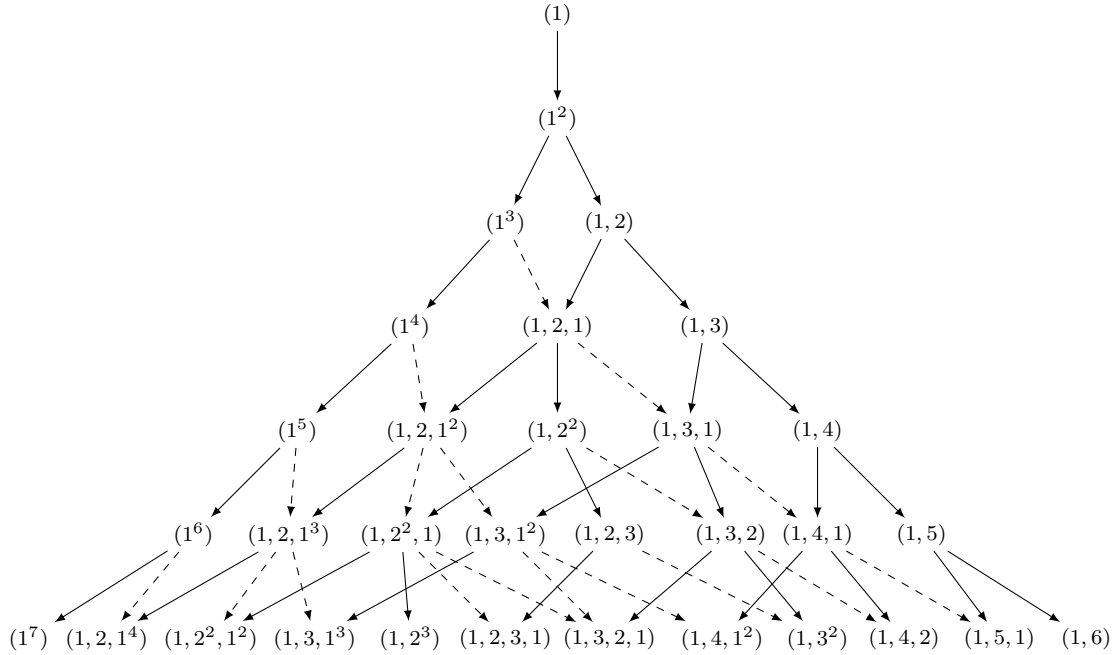


FIGURE 1. The O-sequence graph  $\mathcal{G}$  up to multiplicity 7. The dashed edges are edges of  $\mathcal{G}$  that do not belong to the spanning tree  $\mathcal{T}$ .

In most situations, we will work with O-sequences with fixed multiplicity (i.e. with nodes of  $\mathcal{G}$  at the same distance from the root) or with fixed length. We denote by  $\mathcal{G}_d$  the set of O-sequences of multiplicity  $d$  and by  $\mathcal{G}^s$  the set of O-sequences of length  $s$ .

**Remark 3.4.** As in the spanning tree  $\mathcal{T}$  each vertex is the tail of at most 2 edges, we have that  $|\mathcal{G}_d| < 2|\mathcal{G}_{d-1}|$ . Moreover, since  $|\mathcal{G}_2| = 1$ , by recursion  $|\mathcal{G}_d| < 2^{d-2}$ .

**Lemma 3.5.** (i) The subgraph  $\mathcal{G}^s \subset \mathcal{G}$  is a rooted connected graph with root  $(1^s)$ .

- (ii)  $\mathcal{G}^s$  contains a spanning tree  $\mathcal{T}^s$  with the same root.
- (iii) Denoted by  $d_{\mathcal{G}}^s(\mathbf{h})$  the distance of the node  $\mathbf{h}$  from the root of  $\mathcal{G}^s$ , we have  $d_{\mathcal{G}}^s(\mathbf{h}) = d_{\mathcal{G}}(\mathbf{h}) - (s - 1) = e(\mathbf{h}) - s$ .

*Proof.* We need to show that for any O-sequence  $\mathbf{h} \neq (1^s)$  of length  $s$  there exists another O-sequence of the same length with multiplicity  $e(\mathbf{h}) - 1$ . If  $k = \max\{1 \leq i \leq s - 1 \mid h_i > 1\}$ , then  $\mathbf{h} = (1, h_1, \dots, h_k, 1, \dots, 1)$  and  $\mathbf{h}' = (1, h_1, \dots, h_k - 1, 1, \dots, 1)$  is admissible.  $\square$

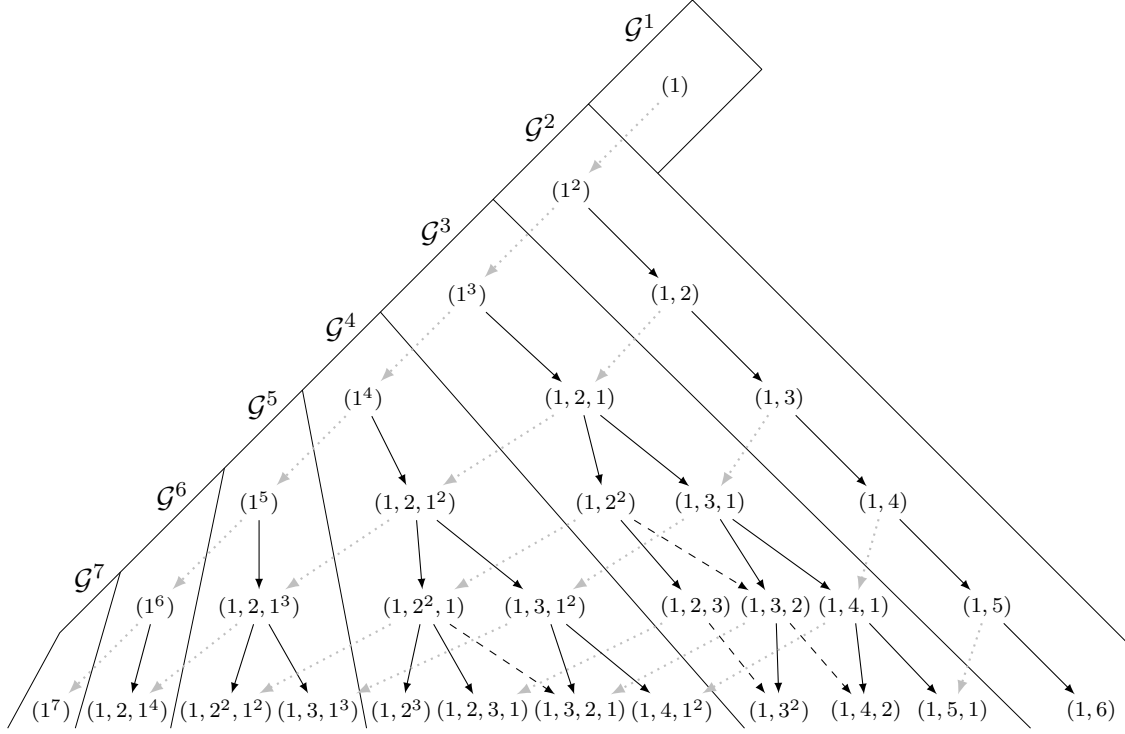


FIGURE 2. The subgraphs  $\mathcal{G}^s$  of the O-sequence graph with fixed length  $s$ . Along the grey dotted edges the length increases, so such edges of  $\mathcal{G}$  do not belong to any subgraph  $\mathcal{G}^s$ . The dashed edges are edges of  $\mathcal{G}^s$  that do not belong to the corresponding spanning tree  $\mathcal{T}^s$ .

The set  $\mathcal{G}_d$  is not a subgraph of  $\mathcal{G}$  as there are no edges of  $\mathcal{G}$  between O-sequences with the same multiplicity. But the edges of  $\mathcal{G}$  induce a natural partial order on  $\mathcal{G}_d$  in the following way.

**Definition 3.6.** Two O-sequences  $\mathbf{h}_1$  and  $\mathbf{h}_2$  in  $\mathcal{G}_d$  are *directly comparable* if there exists  $\mathbf{h}_0 \in \mathcal{G}_{d-1}$  such that  $\mathbf{h}_1 = \mathbf{h}_0 + \mathbf{e}_i$  and  $\mathbf{h}_2 = \mathbf{h}_0 + \mathbf{e}_j$ , i.e.  $\mathbf{h}_1 - \mathbf{h}_2 = \mathbf{e}_i - \mathbf{e}_j$ . On directly comparable O-sequences we consider the order

$$(3.1) \quad \mathbf{h}_1 \prec \mathbf{h}_2 \iff i < j$$

and its transitive closure in  $\mathcal{G}_d$  will be also denoted by  $\prec$ .

The partial order  $\prec$  naturally gives a structure of directed graph to  $\mathcal{G}_d$ . As edges, we consider the elements  $\mathbf{e}_j - \mathbf{e}_i$ ,  $j > i$  such that there exist  $\mathbf{h}, \mathbf{h}' \in \mathcal{G}_d$  and  $\mathbf{h} = \mathbf{h}' + \mathbf{e}_j - \mathbf{e}_i$  (see Figure 3(a)). As before, we would like to define a spanning tree of the graph structure of  $\mathcal{G}_d$  allowing us to examine efficiently the set of O-sequences with fixed multiplicity. The same should be also extended to the set of O-sequences  $\mathcal{G}_d^s$  with fixed multiplicity  $d$  and length  $s$ .

**Proposition 3.7.** (i) The graph  $\mathcal{G}_d$  contains a spanning tree  $\mathcal{T}_d$  with root the O-sequence  $(1, d - 1)$ .



(ii) The subgraph  $\mathcal{G}_d^s$  contains a spanning tree  $\mathcal{T}_d^s$  with root the O-sequence  $(1, d - s + 1, 1^{s-2})$ . Thus,  $\mathcal{G}_d^s$  is also connected.

*Proof.* (i) For each vertex  $\mathbf{h} \in \mathcal{G}_d \setminus \{(1, d - 1)\}$ , the spanning tree  $\mathcal{T}_d$  contains the edge  $\mathbf{e}_{s-1} - \mathbf{e}_1$  going from  $\mathbf{h}' = \mathbf{h} - \mathbf{e}_{s-1} + \mathbf{e}_1$  to  $\mathbf{h}$ , where  $s$  is the length of  $\mathbf{h}$ .

(ii) For each vertex  $\mathbf{h} = (1, h_1, \dots, h_i, 1^{d-\sum_{j=0}^i h_j}) \in \mathcal{G}_d^s \setminus \{(1, d - s + 1, 1^{s-2})\}$  (i.e.  $i > 1$ ), the spanning tree  $\mathcal{T}_d^s$  contains the edge  $\mathbf{e}_i - \mathbf{e}_1$  going from  $\mathbf{h}' = \mathbf{h} - \mathbf{e}_i + \mathbf{e}_1$  to  $\mathbf{h}$ .  $\square$

**Lemma 3.8.** *If  $\mathbf{h}_1 - \mathbf{h}_2 = \mathbf{e}_i - \mathbf{e}_j$ , then  $g(\mathbf{h}_1) = g(\mathbf{h}_2) + (i - 1) - (j - 1) = g(\mathbf{h}_2) + i - j$ . Then, the order induced on  $\mathcal{G}_d$  by the total order on  $\mathbb{N}$  through the map  $g : \mathcal{G}_d \rightarrow \mathbb{N}$  is a refinement of the partial order  $\prec$ .*

*Proof.* The first statement follows applying twice Lemma 3.2(ii). Then, we obtain

$$\mathbf{h}_1 \prec \mathbf{h}_2 \iff i < j \implies g(\mathbf{h}_1) < g(\mathbf{h}_2). \quad \square$$

**Example 3.9.** For  $\mathbf{h}_1 = (1, 4, 10, 16, 22, 29)$ , we get  $g(\mathbf{h}_1) = 10 + 16 \cdot 2 + 22 \cdot 3 + 29 \cdot 4 = 224$ . If we take  $i = 5$  and  $j = 3$ , then we obtain  $\mathbf{h}_2 = \mathbf{h}_1 + \mathbf{e}_5 - \mathbf{e}_3 = (1, 4, 10, 15, 22, 30)$  and  $g(\mathbf{h}_2) = g(\mathbf{h}_1) + 5 - 3 = 226$ . If we take  $i = 4$  and  $j = 5$ , then we get  $\mathbf{h}_3 = \mathbf{h}_1 + \mathbf{e}_4 - \mathbf{e}_5 = (1, 4, 10, 16, 23, 28)$  and  $g(\mathbf{h}_3) = g(\mathbf{h}_1) + 4 - 5 = 223$ .

Now, we can state the strategy of a general algorithm for searching aCM genera. Depending on the constraints on multiplicity and length we consider, we will choose the corresponding set of O-sequences and, more precisely, the associated spanning tree  $\tilde{\mathcal{T}}$ . Then we will perform a depth-first search on the tree using a LIFO (Last In First Out) procedure of visit of the vertices. Assume that at some moment of the search, we stored in a list (or a stack) the vertices of which we already know the existence (we have visited their parents) but which we have not visited yet. We visit the first vertex  $\mathbf{h}$  in the list (or the top of the stack). There are three possible actions to be performed:

- A. if  $g(\mathbf{h})$  is equal to the genus we are looking for, we end the visit returning the O-sequence  $\mathbf{h}$ ;
- B. if  $g(\mathbf{h})$  is greater than the genus we are looking for, as the genus increases along the edges (Lemma 3.2(ii) and Lemma 3.8), we can avoid to visit the tree of descendants of  $\mathbf{h}$ ;
- C. if  $g(\mathbf{h})$  is smaller than the genus we are looking for, we need to visit the tree of descendants of  $\mathbf{h}$ , so we add the children of  $\mathbf{h}$  in the tree  $\tilde{\mathcal{T}}$  at the beginning of the list (or at the top of the stack) containing the vertices still to be visited.

Now, we extend the order introduced in Definition 3.6 to the following total order on  $\mathcal{G}_d^s$ , for every  $s \in \{2, \dots, d - 1\}$ .

**Definition 3.10.** Given two O-sequences  $\mathbf{h} = (1, h_1, \dots, h_{s-1})$  and  $\mathbf{h}' = (1, h'_1, \dots, h'_{s-1})$  of  $\mathcal{G}_d^s$ , we denote by  $<$  the total order on  $\mathcal{G}_d^s$  such that  $\mathbf{h} < \mathbf{h}'$  if  $h_\ell < h'_\ell$ , where  $\ell := \max\{j : h_j \neq h'_j\}$ .

Although the order  $<$  of Definition 3.10 extends the order  $\prec$  of Definition 3.6, the conclusion of Lemma 3.8 cannot be extended. Anyway, we obtain the following result.

**Theorem 3.11.** *Let  $\mathbf{h} = (1, h_1, \dots, h_{s-1})$  and  $\mathbf{k} = (1, k_1, \dots, k_{s-1})$  be two O-sequences of  $\mathcal{G}_d^s$ . If  $\mathbf{k} < \mathbf{h}$  and  $g(\mathbf{k}) > g(\mathbf{h})$ , then there is an O-sequence  $\tilde{\mathbf{h}} \in \mathcal{G}_d^s$  such that  $\tilde{\mathbf{h}} > \mathbf{h}$  and  $g(\tilde{\mathbf{h}}) > g(\mathbf{k})$ . Thus,  $\max\{g(\tilde{\mathbf{h}}) \mid \tilde{\mathbf{h}} \in \mathcal{G}_d^s\} = g(\max(\mathcal{G}_d^s))$ .*

*Proof.* We can assume  $s - 1 = \max\{j : h_j \neq k_j\}$ , hence  $h_{s-1} > k_{s-1}$  because  $\mathbf{h} > \mathbf{k}$ . By the hypotheses, we have

$$g(\mathbf{h}) = \sum_{j=1}^{s-2} (j-1)h_j + (s-2)h_{s-1} < \sum_{j=1}^{s-2} (j-1)k_j + (s-2)k_{s-1} = g(\mathbf{k})$$

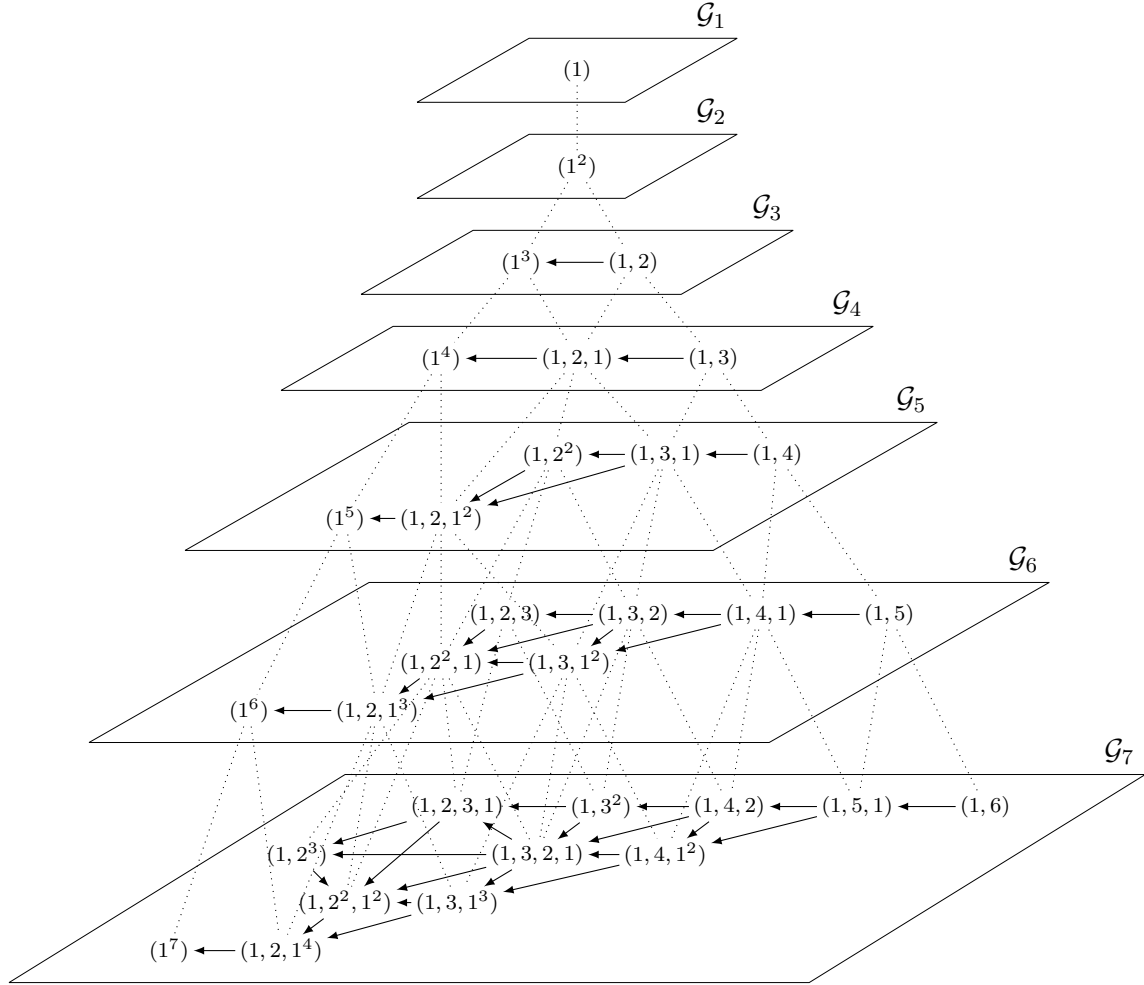
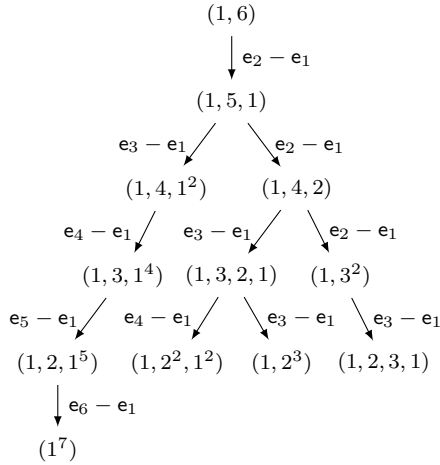
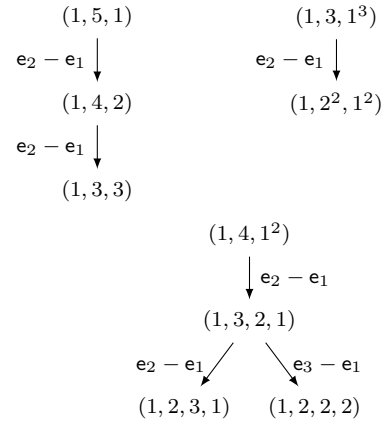
(A) The order relations among directly comparable elements of  $\mathcal{G}_d$ ,  $d = 1, \dots, 7$ .(B) The spanning tree  $\mathcal{T}_7$  of  $\mathcal{G}_7$ .(C) The spanning trees  $\mathcal{T}_7^s$  of  $\mathcal{G}_7^s$  for  $s = 3, 4, 5$ . For  $s = 2, 6, 7$ , the graph  $\mathcal{G}_7^s$  has a unique vertex.

FIGURE 3. Graph descriptions of O-sequences with fixed multiplicity.

which implies there exists the integer  $t := \max\{j \in \{2, \dots, s-2\} : h_j < k_j\}$  and so

$$(3.2) \quad \begin{array}{ccccccc} (1, & h_1, & \dots, & h_t, & h_{t+1}, & \dots, & h_{s-2}, & h_{s-1}) \\ & & & \wedge & \vee & & \vee & \vee \\ (1, & k_1, & \dots, & k_t, & k_{t+1}, & \dots, & k_{s-2}, & k_{s-1}) \end{array}$$



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**Algorithm 1** The algorithm for searching for aCM genera with given constraints on the multiplicity and the length of the O-sequences. A trial version of this algorithm is available at [www.personalweb.unito.it/paolo.lella/HSC/Finite\\_O-sequences\\_and\\_ACM\\_genus.html](http://www.personalweb.unito.it/paolo.lella/HSC/Finite_O-sequences_and_ACM_genus.html)

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1: GENUSSEARCH( $g, \tilde{\mathcal{T}}$ )

**Require:**  $g$ , a non-negative integer.

**Require:**  $\tilde{\mathcal{T}}$ , a spanning tree chosen among  $\mathcal{T}$ ,  $\mathcal{T}_d$ ,  $\mathcal{T}^s$  and  $\mathcal{T}_d^s$ .

**Ensure:** an O-sequence  $\mathbf{h}$  such that  $g(\mathbf{h}) = g$  (if it exists).

2:  $\text{stack} := \{\text{ROOT}(\tilde{\mathcal{T}})\};$

3: **while**  $\text{stack} \neq \emptyset$  **do**

4:    $\mathbf{h} := \text{REMOVEFIRST}(\text{stack});$

5:   **if**  $g(\mathbf{h}) = g$  **then**

6:     **return**  $\mathbf{h}$ ;

7:   **else if**  $g(\mathbf{h}) < g$  **then**

8:      $\text{ADDFIRST}(\text{stack}, \text{CHILDREN}(\mathbf{h}, \tilde{\mathcal{T}}));$

9:   **end if**

10: **end while**

---

that is

$$\begin{cases} h_t < k_t, \\ h_i \geq k_i, & t+1 \leq i \leq s-2, \\ h_{s-1} > k_{s-1}. \end{cases}$$

Note that  $k_t^{(t)} \geq h_t^{(t)} \geq h_{t+1} \geq k_{t+1}$ . Hence, we can consider the O-sequence  $\mathbf{h}' := \mathbf{k} - b\mathbf{e}_t + \sum_{j=t+1}^{s-1} c_j \mathbf{e}_j$ , where

$$b = \min \left\{ k_t - h_t, \sum_{j=t+1}^{s-1} h_j - k_j \right\} \quad \text{and} \quad c_j = \min \left\{ h_j - k_j, b - \sum_{i=t+1}^{j-1} c_i \right\}$$

and  $h'_j \leq h_j$  for every  $j > t$ .

The corresponding genus of  $\mathbf{h}'$  is

$$g(\mathbf{h}') = g(\mathbf{k}) - (t-1)b + \sum_{j=t+1}^{s-1} (j-1)c_j > g(\mathbf{k}) > g(\mathbf{h}).$$

If needed, replacing the O-sequence  $\mathbf{k}$  by  $\mathbf{h}'$  and repeating the same argument as before, we obtain an O-sequence  $\mathbf{h}'$  with  $h'_j = h_j$  for every  $j > t$  and  $g(\mathbf{h}') > g(\mathbf{h})$ . If  $\mathbf{h}' < \mathbf{h}$ , we can repeat the same argument as before until we obtain an O-sequence  $\bar{\mathbf{h}}$  with  $\bar{h}_j = h_j$  for every  $j > t$  and  $\bar{h}_t \geq h_t + 1$ .  $\square$

**Example 3.12.** (a) Consider the two O-sequences  $\mathbf{h} = (1, 6, 4, 2, 1)$  and  $\mathbf{k} = (1, 4, 7, 1, 1)$  of  $\mathcal{G}_{14}^5$ . We have  $\mathbf{h} > \mathbf{k}$  and  $11 = g(\mathbf{h}) < g(\mathbf{k}) = 12$  as in the hypotheses of Theorem 3.11. In this case, we obtain  $t = 2$ ,  $b = \min\{3, 1\} = 1$ ,  $c_3 = \min\{1, 3\} = 1$  and  $c_4 = \min\{0, 2\} = 0$ , so that the O-sequence  $\bar{\mathbf{h}}$  is  $\mathbf{k} - \mathbf{e}_2 + \mathbf{e}_3 = (1, 4, 6, 2, 1)$  with genus  $g(\bar{\mathbf{h}}) = 13 > g(\mathbf{k})$  and  $\bar{\mathbf{h}} > \mathbf{h}$ .

(b) Consider the two O-sequences  $\mathbf{h} = (1, 13, 3, 3, 3)$  and  $\mathbf{k} = (1, 6, 13, 2, 1)$  of  $\mathcal{G}_{23}^5$ . We have  $\mathbf{h} > \mathbf{k}$  and  $18 = g(\mathbf{h}) < g(\mathbf{k}) = 20$ . Applying Theorem 3.11, as  $t = 2$ ,  $b = \min\{10, 3\} = 3$ ,  $c_3 = \min\{1, 10\} = 1$  and  $c_4 = \min\{2, 9\} = 2$ , we determine  $\bar{\mathbf{h}} = \mathbf{k} - 3\mathbf{e}_2 + \mathbf{e}_3 + 2\mathbf{e}_4 = (1, 6, 10, 3, 3) > \mathbf{h}$  and  $g(\bar{\mathbf{h}}) = 18 + 2 + 3 = 21 > g(\mathbf{k})$ .

#### 4. COMBINATORIAL RANGES FOR ACM GENERA

Recall that, if  $I \subset S$  is the defining (saturated) ideal of an aCM curve  $C$  of degree  $d$ , then  $d$  is the multiplicity of the  $K$ -algebra  $S/I$ . Thus, by Remark 2.3, it is sufficient to

consider the finite O-sequences of multiplicity  $d$ . From now we assume  $d > 2$ , as  $\mathcal{G}_d$  has only one element for  $d \in \{1, 2\}$ .

Looking at the graph  $\mathcal{G}_d$ , we can immediately get another proof of the well known fact that the arithmetic genus  $g$  of an aCM curve  $C$  must belong to the range  $R_d := \{0, \dots, \binom{d-1}{2}\}$ . In fact, the genus is non-negative by Theorem 1.7 (see also Lemma 2.4) and, as  $(1^d) \succ \mathbf{h}$  for every  $\mathbf{h} \in \mathcal{G}_d$ , by Lemma 3.8 we have that  $g((1^d)) = \binom{d-1}{2}$  is the maximal possible arithmetic genus of projective curves of degree  $d$  (see [4, Theorem 3.1]).

Moreover, as the vertex  $(1^d)$  is the tail of no edges and the head of a unique edge with tail  $(1, 2, 1^{d-3})$ , the O-sequence  $(1, 2, 1^{d-3})$  is greater than any other O-sequence  $\mathbf{h} \in \mathcal{G}_d \setminus \{(1^d)\}$  with respect to  $\prec$ . Hence, unless  $C$  is a plane curve, the genus of an aCM curve is upper bounded by  $g((1, 2, 1^{d-3})) = \binom{d-2}{2}$  (cf. [4, Theorem 3.3]).

From now on, for convenience, we denote by  $[a, b]$  the set of integers  $\{n \in \mathbb{N} \mid a \leq n \leq b\}$ .

Our aim is to detect what integers in  $R_d = [0, \binom{d-1}{2}]$  are the arithmetic genera of aCM curves of degree  $d$ . To this aim, in the range  $R_d$  we will find smaller ranges, taking into account the length of the O-sequences.

We denote by  $G_d$  the set of all the arithmetic genera of the aCM curves of degree  $d$  and by  $G_d^s$  the set of all the arithmetic genera of the aCM curves of degree  $d$  with  $h$ -vector of length  $s$ . Moreover, we denote by  $\mathbf{h}^s(d)$  the minimum and by  $\mathbf{h}_s(d)$  the maximum of  $\mathcal{G}_d^s$  with respect to the order  $>$ . We let  $\mathbf{g}^s(d) := g(\mathbf{h}^s(d))$  and  $\mathbf{g}_s(d) := g(\mathbf{h}_s(d))$ .

**Lemma 4.1.** *For every  $s \in \{2, \dots, d\}$ ,*

$$(4.1) \quad \mathbf{h}^s(d) := (1, d-s+1, 1^{s-2}) \text{ and } \mathbf{g}^s(d) = \binom{s-1}{2}.$$

*For every  $s \in \{\lfloor \frac{d}{2} \rfloor + 1, \dots, d\}$ ,*

$$(4.2) \quad \mathbf{h}_s(d) := (1, 2^{d-s}, 1^{2s-d-1}) \text{ and } \mathbf{g}_s(d) = \binom{s-1}{2} + \binom{d-s}{2}.$$

*Proof.* The conclusions of this statement follow by the definition of the order  $<$  on  $\mathcal{G}_d^s$  and by formula (2.1).  $\square$

By Lemma 4.1, it is evident that the genus  $\mathbf{g}^s(d)$  does not depend on the value of  $d$ . Hence, from now we will denote it only by  $\mathbf{g}^s$ .

**Proposition 4.2.** *For every  $d \geq s \geq 2$ ,  $\min(G_d^s) = \mathbf{g}^s$  and  $\max(G_d^s) = \mathbf{g}_s(d)$ .*

*Proof.* The equality  $\min(G_d^s) = \mathbf{g}^s$  follows from the fact that the O-sequence  $\mathbf{h}^s(d)$  is the root of  $\mathcal{T}_d^s$ , i.e. smaller than any other element of  $\mathcal{G}_d^s$  with respect to  $<$ , and by Lemma 3.8. The equality  $\max(G_d^s) = \mathbf{g}_s(d)$  follows immediately from Theorem 3.11.  $\square$

**Definition 4.3.** For every  $d \geq s \geq 2$ , the set of integers between  $\mathbf{g}^s$  and  $\mathbf{g}_s(d)$  is called  $(d, s)$ -range and denoted by  $R_d^s$ , i.e.  $R_d^s := \{\alpha \in \mathbb{N} \mid \binom{s-1}{2} \leq \alpha \leq \mathbf{g}_s(d)\}$ .

**Corollary 4.4.** *For every  $d \geq s \geq 2$ , the arithmetic genus of an aCM curve  $C$  of degree  $d$  having  $h$ -vector of length  $s$  belongs to the range  $R_d^s$ .*

**Remark 4.5.** For every  $d > 2$ , we have

- (i)  $\mathbf{g}^d = \binom{d-1}{2} = \mathbf{g}_d(d)$  and  $\mathbf{g}^{d-1} = \binom{d-2}{2} = \mathbf{g}_{d-1}(d)$
- (ii)  $\mathbf{g}^{s-1} < \mathbf{g}^s$  and  $\mathbf{g}^s - \mathbf{g}^{s-1} = s - 2$ , for every  $s \in \{2, \dots, d\}$

Note that Lemma 4.1 gives a complete computation of the minimal genera, but a partial computation of the maximal ones. Anyway, we can compute the upper bound of the ranges  $R_d^s$  recursively looking at the graph  $\mathcal{G}^s$ . The idea is to deduce the maximal genus of  $R_d^s$  knowing the O-sequence realizing the maximal genus of  $R_{d-1}^s$ . The procedure is the following:

- for  $d < s$ ,  $R_d^s$  is empty and for  $d = s$  we have a unique O-sequence  $(1^s)$  corresponding to a plane curve of degree  $s$ , i.e. with genus  $\binom{s-1}{2}$ , so that  $R_d^s = \{(\binom{s-1}{2})\}$ .
- Let us assume that we know the maximal genus  $g$  of the range  $R_{d-1}^s$  and the O-sequence  $\mathbf{h}$  such that  $g = \sum_{j=2}^{s-1} h_j(j-1)$ . To compute the upper bound of  $R_d^s$ , we look at the node  $\mathbf{h} \in \mathcal{G}^s$  and we follow the edge corresponding at the increment of the value of  $\mathbf{h}$  with highest index (see Figure 4 for an example).

By the above procedure, besides the results of Lemma 4.1 we find also the value of  $\mathbf{g}_s(d)$ , for every  $s \in \{2, \dots, \lfloor \frac{d}{2} \rfloor\}$ . Indeed, we prove the following result.

**Proposition 4.6.** *For every  $d \geq s \geq 3$ , if  $\iota$  is the highest index such that  $\mathbf{h}_s(d-1) + \mathbf{e}_\iota$  is an O-sequence in  $\mathcal{G}_d^s$ , then  $\mathbf{h}_s(d-1) + \mathbf{e}_\iota = \mathbf{h}_s(d)$ . In particular,  $\mathbf{g}_s(d) = \mathbf{g}_s(d-1) + \iota - 1$ .*

*Proof.* For convenience, we denote by  $\mathbf{h}$  the O-sequence  $\mathbf{h}_s(d-1)$ . Hence, we have

$$(4.3) \quad \begin{aligned} \mathbf{h} &= (1, \dots, h_\iota, h_\iota^{(\iota)}, h_{\iota+1}^{(\iota+1)}, \dots, h_{s-2}^{(s-2)}) \\ \mathbf{h} + \mathbf{e}_\iota &= (1, \dots, h_\iota + 1, h_\iota^{(\iota)}, h_{\iota+1}^{(\iota+1)}, \dots, h_{s-2}^{(s-2)}). \end{aligned}$$

We want to prove  $\mathbf{h} + \mathbf{e}_\iota = \mathbf{h}_s(d)$ . On the contrary, assume that there is an O-sequence  $\mathbf{h}' \in \mathcal{G}_{d-1}^s$  such that  $\mathbf{h}' + \mathbf{e}_\ell > \mathbf{h} + \mathbf{e}_\iota$  for some index  $\ell$  for which  $\mathbf{h}' + \mathbf{e}_\ell \in \mathcal{G}_d^s$ . Then, we have  $\iota < \ell$ , because  $\mathbf{h}' < \mathbf{h}$  by hypothesis. Hence, if

$$(4.4) \quad \mathbf{h}' + \mathbf{e}_\ell = (1, \dots, h'_\iota, \dots, h'_\ell + 1, h'_{\ell+1}, \dots, h'_{s-1}),$$

then  $h'_{\ell+1} = h_{\ell+1} = h_\ell^{(\ell)}, \dots, h'_{s-1} = h_{s-1} = h_\ell^{(s-2)}$  and  $h'_\ell = h_\ell$ , because  $(h'_\ell)^{(\ell)} \geq h'_{\ell+1} = h_{\ell+1} = h_\ell^{(\ell)}$  and because  $\mathbf{h}' < \mathbf{h}$ . Moreover, we must have  $h'_{\ell-1} > h_{\ell-1}$ , otherwise  $\mathbf{h}' + \mathbf{e}_\ell$  would be non-admissible. Thus, we get the contradiction  $\mathbf{h}' > \mathbf{h}$ , being  $h'_{\ell-1} > h_{\ell-1}$ . The last assertion follows by Proposition 4.2 and formula (2.1).  $\square$

**Remark 4.7.** An other description of the maximal genus of a range  $R_d^s$  can be given in terms of minimal Hilbert functions with a constant Hilbert polynomial and a fixed regularity (see [7, Examples 4.6 and 4.8] and [3]). Anyway, the combinatorial description we provide here arises in a very natural way and gives more information, at least from a computational point of view.

## 5. UNATTAINABLE ACM GENERA IN $R_d$

Recall that we are denoting by  $R_d$  the range  $[0, \binom{d-1}{2}]$  and that  $G_d \subset R_d$ .

**Definition 5.1.** An integer in  $R_d \setminus G_d$  is called *gap* in  $R_d$ .

**Example 5.2.** The integers in the range  $[\binom{d-2}{2} + 1, \binom{d-1}{2} - 1]$  are gaps in  $R_d$ , as seen at the beginning of Section 4. More generally, every integer of  $R_d$  not contained in any  $(d, s)$ -range is a gap.

We begin this section looking for consecutive  $(d, s)$ -ranges that are *separated*, i.e. ranges  $R_d^s$  and  $R_d^{s+1}$  such that  $\mathbf{g}^{s+1} - \mathbf{g}_s(d) > 1$ .

**Proposition 5.3.** *Fixed any  $d > 2$ ,*

$$\mathbf{g}_s(d) < \mathbf{g}^{s+1} - 1 \iff \frac{2d+1-\sqrt{8d-15}}{2} < s \leq d-1.$$

*Thus, for  $\frac{2d+1-\sqrt{8d-15}}{2} < s \leq d-1$  the integers in  $[\mathbf{g}_s(d) + 1, \mathbf{g}^{s+1} - 1]$  are gaps in  $R_d$ .*

*Proof.* Assume  $s \geq \lfloor \frac{d}{2} \rfloor + 1$ . We can apply Lemma 4.1, so that

$$\mathbf{g}_s(d) < \mathbf{g}^{s+1} - 1 \iff \binom{s-1}{2} + \binom{d-s}{2} < \binom{s}{2} - 1.$$

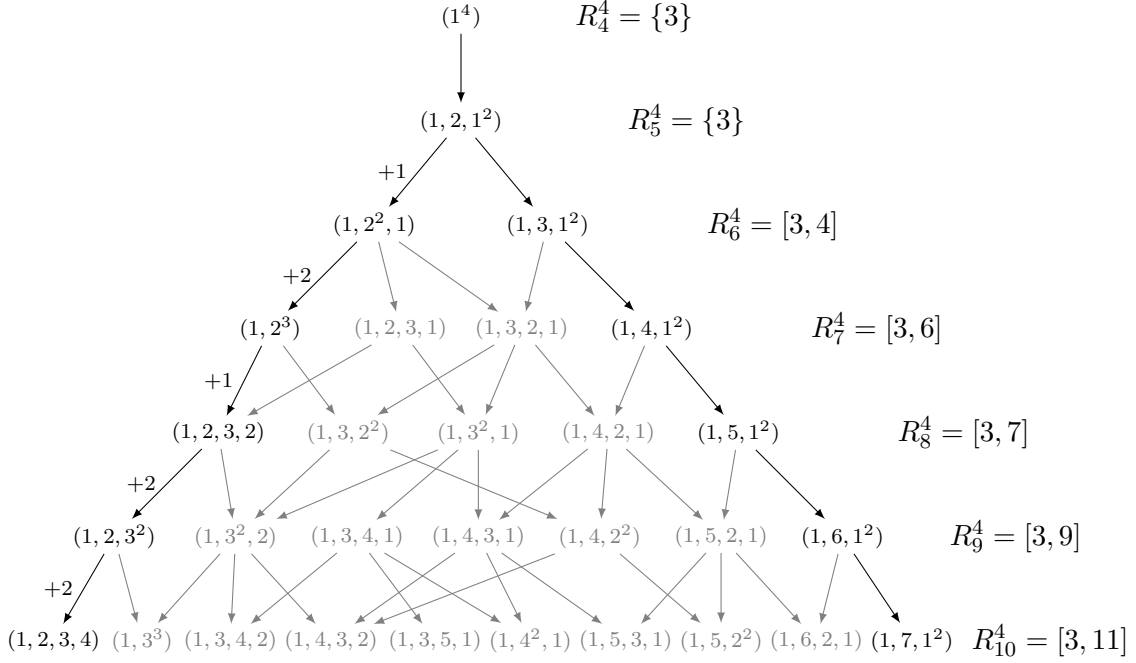


FIGURE 4. The ranges  $R_d^4$  for  $d = 4, \dots, 10$ . In the picture, the edges on the left are labeled with the corresponding increase of the genus.

We have

$$g_s(d) - g^{s+1} + 1 = \frac{s^2 - (2d+1)s + d^2 - d + 4}{2} < 0 \Rightarrow \frac{2d+1-\sqrt{8d-15}}{2} < s < \frac{2d+1+\sqrt{8d-15}}{2},$$

thus  $g_s(d) < g^{s+1} - 1$  if and only if  $\frac{2d+1-\sqrt{8d-15}}{2} < s \leq d-1$ , because  $\frac{2d+1-\sqrt{8d-15}}{2} > \lfloor \frac{d}{2} \rfloor$ ,  $\frac{2d+1+\sqrt{8d-15}}{2} > d-1$  and  $\frac{2d+1-\sqrt{8d-15}}{2} > d-1$  implies  $d < 3$ .

To prove that there are no other pairs of separated ranges, we notice that  $g_s(d) \geq g^{s+1} - 1$  implies  $g_{s-1}(d) \geq g^s - 1$ . Indeed, as  $g^s = g^{s+1} - (s-1)$  and  $g_s(d) \leq g_{s-1}(d) + (s-2)$ , we have

$$g_{s-1}(d) - g^s + 1 \geq g_s(d) - (s-2) - g^{s+1} + (s-1) + 1 > g_s(d) - g^{s+1} + 1 \geq 0. \quad \square$$

**Example 5.4.** For  $d = 17$ , we find  $\frac{2d+1-\sqrt{8d-15}}{2} = 12$ . Hence, the ranges  $R_{17}^s$  and  $R_{17}^{s+1}$  are separated for  $12 < s < 17$ . Precisely

$$\begin{aligned}
g_{12}(17) = 65, \quad g^{13} = 66 &\rightarrow \text{no gaps} \\
g_{13}(17) = 72, \quad g^{14} = 78 &\rightarrow \text{gaps } [73, 77] \\
g_{14}(17) = 81, \quad g^{15} = 91 &\rightarrow \text{gaps } [82, 90] \\
g_{15}(17) = 92, \quad g^{16} = 105 &\rightarrow \text{gaps } [93, 104] \\
g_{16}(17) = 105, \quad g^{17} = 120 &\rightarrow \text{gaps } [106, 119] \text{ (see Example 5.2).}
\end{aligned}$$

**Example 5.5.** For every  $d \leq 11$ , the gaps in  $R_d$  are only those described in Proposition 5.3. For  $d = 12$ , in addition to the gaps described in Proposition 5.3, by examining all the Artinian admissible O-sequences of multiplicity 12, we find a unique further gap  $\bar{g} = 26$ , which belongs only to the range  $R_{12}^8 = [21, 28]$ .

**Example 5.6.** By a direct computation of the Artinian admissible O-sequences, we note that for  $d = 15$  the integer  $\bar{g} = 25$  belongs to the ranges  $R_{15}^6$  and  $R_{15}^5$ . Meanwhile for each  $h \in R_{15}^5$  we have  $g(h) \neq 25$ , there is  $\bar{h} = (1, 3, 3, 4, 2, 2) \in R_{15}^6$  such that  $g(\bar{h}) = 25$ .

Example 5.6 suggests the following definition.

**Definition 5.7.** An integer in the range  $R_d^s$  is called a *hole* of the range  $R_d^s$  if it is not the arithmetic genus of an aCM curve  $C$  of degree  $d$  with  $h$ -vector of length  $s$ .

**Remark 5.8.** There are holes that are not gaps. For instance, the integer 25 is not a gap in  $R_{15}$ , although it is a hole of  $R_{15}^5$ , as we have seen in the Example 5.6. Instead, the integer 26 is a hole of  $R_{12}^8$  and is also a gap in  $R_{12}$ , as we have seen in Example 5.5.

Notice that for  $s = d - 1, d - 2, d - 3$  there are no holes, as

- $\mathcal{G}_d^{d-1}$  contains a unique O-sequence  $(1, 2, 1^{d-3})$ ;
- $\mathcal{G}_d^{d-2}$  contains two O-sequences:  $(1, 2, 2, 1^{d-5})$  and  $(1, 3, 1^{d-4})$  corresponding to genera  $\binom{d-3}{2} + 1$  and  $\binom{d-3}{2}$ ;
- $\mathcal{G}_d^{d-3}$  contains four O-sequences:  $(1, 2, 2, 2, 1^{d-7})$ ,  $(1, 2, 3, 1^{d-6})$ ,  $(1, 3, 2, 1^{d-6})$  and  $(1, 4, 1^{d-5})$  corresponding to genera  $\binom{d-4}{2} + 3$ ,  $\binom{d-4}{2} + 2$ ,  $\binom{d-4}{2} + 1$  and  $\binom{d-4}{2}$ .

Now, we detect some values of  $d$  and  $s$  for which in the ranges  $R_d^s$  there exist certain special gaps, pointing out some particular holes which are also gaps, because they belong to *parts* of  $(d, s)$ -ranges not overlapping each other.

**Lemma 5.9.** *Let  $d$  and  $s$  be such that  $7 \leq \lfloor \frac{d}{2} \rfloor + 1 \leq s \leq d - 4$ . The integers  $\mathbf{g}_s(d) - (d - s - 3), \dots, \mathbf{g}_s(d) - 1$  are holes in the range  $R_d^s$ .*

*Proof.* By Lemma 4.1 and Proposition 4.2, the maximum genus  $\mathbf{g}_s(d)$  in  $R_d^s$  is realized by the O-sequence  $\mathbf{h}_s(d) = (1, 2^{d-s}, 1^{2s-d-1})$ . In the graph  $\mathcal{G}_d^s$ , this vertex is the tail of no edges, whereas it is the head of two edges coming from  $(1, 3, 2^{d-s-2}, 1^{2s-d})$  and  $(1, 2, 3, 2^{d-s-3}, 1^{2s-d})$ . Hence, for each  $\mathbf{h} \in \mathcal{G}_d^s \setminus \{\mathbf{h}_s(d)\}$

$$\begin{aligned} g(\mathbf{h}) &\leq \max \{g((1, 3, 2^{d-s-2}, 1^{2s-d})), g((1, 2, 3, 2^{d-s-3}, 1^{2s-d}))\} \\ &= \max \{\mathbf{g}_s(d) + 1 - (d - s), \mathbf{g}_s(d) + 2 - (d - s)\} = \mathbf{g}_s(d) - (d - s - 2) \end{aligned}$$

by Lemma 3.8. □

All the holes described in the previous lemma are surely gaps if we consider  $s > \frac{2d+1-\sqrt{8d-15}}{2}$  as in Proposition 5.3. Indeed, these holes do not belong to any other range. Now we give a sharp condition for which one of these holes is a gap.

**Proposition 5.10.** *In the hypotheses of Lemma 5.9, for every  $i = 1, \dots, d - s - 3$ , the hole  $\mathbf{g}_s(d) - i$  is a gap if  $s - 1 - \binom{d-s}{2} + i > 0$ . More precisely,*

- (i) *the highest hole  $\mathbf{g}_{d-4}(d) - 1 = \frac{d(d-11)}{2} + 20$  is always a gap;*
- (ii) *every hole described in Lemma 5.9 is a gap if  $s > \frac{2d-1-\sqrt{8d-31}}{2}$ .*

*Proof.* The hole  $\mathbf{g}_s(d) - i$  is a gap if  $\mathbf{g}_s(d) - i < \mathbf{g}^{s+1}$ , i.e.

$$\binom{s}{2} - \binom{s-1}{2} - \binom{d-s}{2} + i = s - 1 - \binom{d-s}{2} + i > 0.$$

The proof of (i) and (ii) is a direct computation. □

**Example 5.11.** By Proposition 5.10, we find the following gaps in  $R_{28}$ : the gap 258 which belongs only to the range  $R_d^{24}$ , the two gaps 240 and 239 which belong only to the range  $R_d^{23}$ , the three gaps 224, 223 and 222, which belong only to the range  $R_d^{22}$ , and also the three gaps 207, 208 e 209 which belong only to the range  $R_{28}^{21}$ . Anyway, by a direct computation of the O-sequences, we check that there is also the gap 188, that is the minimal one.

6. COMPUTATION OF THE ACM GENERA FOR CURVES OF DEGREE  $d$ 

Proposition 5.10 gives a characterization of the gaps in  $R_d$  that belong to the *last part* of a  $(d, s)$ -range. We did not find analogous conditions for gaps that belong to the *first part* of a  $(d, s)$ -range. In particular, it seems hard to give a characterization of the minimal gap. Hence, we will look for an algorithmic method to recognize the gaps in  $R_d$ . We want to compute all the arithmetic genera of the aCM curves of degree  $d$ , avoiding to construct all the finite O-sequences of multiplicity  $d$ . To this aim, we exploit a sort of *continuity* in the generation of the arithmetic genera. We denote by  $G_d + a$  the set of all arithmetic genera of the aCM curves of degree  $d$  augmented by a non-negative integer  $a$ .

**Lemma 6.1.**  $G_d \supseteq G_{d-1} \cup (G_{d-2} + 1) \cup \cdots \cup (G_k + \binom{d-k}{2}) \cup \cdots \cup (G_1 + \binom{d-1}{2})$ .

*Proof.* Let  $(1, h_1, \dots, h_{s-1})$  be an O-sequence of multiplicity  $k < d$  corresponding to an aCM arithmetic genus  $g$ , and suppose  $h_i^{(i)} > h_{i+1}$ , for some  $i \in \{1, \dots, s-2\}$ . Hence we can consider the finite O-sequence  $(1, h_1, \dots, h_{i+1} + 1, \dots, h_{s-1})$  of multiplicity  $k+1$ , corresponding to the genus  $g+i$ . Then, we can take also the finite O-sequence  $(1, h_1, \dots, h_{i+1} + 1, h_{i+2} + 1, \dots, h_{s-1})$  of multiplicity  $k+2$ , corresponding to the genus  $g+i+(i+1)$ , and so on. If we perform this construction from  $i=1$  until  $d-k$ , we obtain the thesis.  $\square$

**Remark 6.2.** By the proof of Lemma 6.1, we can observe that the arithmetic genera determined by the O-sequences  $(1, h_1, \dots, h_{s-1})$  with  $h_i \geq h_{i+1}$ , for every  $0 < i < s-1$ , are included in those detected by Lemma 6.1. For example, we obtain:

$$G_1 = G_2 = \{0\}$$

$$G_3 = G_2 \cup (G_1 + 1) = \{0, 1\}$$

$$G_4 = G_3 \cup (G_2 + 1) \cup (G_1 + 3) = \{0, 1, 3\}$$

$$G_5 = G_4 \cup (G_3 + 1) \cup (G_2 + 3) \cup (G_1 + 6) = \{0, 1, 2, 3, 6\}$$

$$G_6 = G_5 \cup (G_4 + 1) \cup (G_3 + 3) \cup (G_2 + 6) \cup (G_1 + 10) = \{0, 1, 2, 3, 4, 6, 10\}$$

$$G_7 \supset G_6 \cup (G_5 + 1) \cup (G_4 + 3) \cup (G_3 + 6) \cup (G_2 + 10) \cup (G_1 + 15) = \{0, 1, 2, 3, 4, 6, 7, 10, 15\}$$

and, for the multiplicity  $d=7$ , we lose the arithmetic genus  $g=5$  which corresponds to the finite O-sequence  $(1, 2, 3, 1)$ .

Now, we exploit Lemma 6.1 obtaining the computation of large sets of aCM genera, for every  $d \geq 18$ . To this aim, we need to define an increasing sequence  $\{m_d\}_{d \geq 1}$  by the following recursive procedure:

- if  $d=1$  then  $M := 0$ ; if  $d > 1$ , then we take  $M := m_{d-1}$  and, for every  $2 \leq h \leq d-1$ , if  $M \geq \binom{h}{2} - 1$  then we set  $M := \max\{M, m_{d-h} + \binom{h}{2}\}$ ;
- $m_d := M$ .

**Theorem 6.3** (Continuity). *For all  $d \geq 1$ , every integer in  $\{0, \dots, m_d\}$  is the arithmetic genus of an aCM curve of degree  $d$  and  $m_d \geq g^{\lceil \frac{d}{2} \rceil + 2}$ , for every  $d \geq 18$ .*

*Proof.* The first affirmation holds by Lemma 6.1 and by the definition of  $m_d$ . For the second affirmation, it is enough to consider odd degrees  $d$  and to see Table 1 for  $18 \leq d \leq 36$ . If  $d \geq 37$ , let  $s := \lceil \frac{d}{2} \rceil + 2$ . By construction and by induction, we know that  $m_d \geq m_{d-1} \geq g^{\lceil \frac{d-1}{2} \rceil + 2} = \binom{s-2}{2}$ . Hence, by the definition of  $m_d$  we get

$$m_d \geq \max\{m_{d-1}, m_{d-(s-2)} + \binom{s-2}{2}\}.$$

Being  $d$  odd, we have  $d - (s-2) = d - \lceil \frac{d}{2} \rceil = \lceil \frac{d}{2} \rceil - 1 = s-3 \geq 18$ . Thus, by induction we obtain  $m_d \geq \binom{\lceil \frac{s-3}{2} \rceil + 1}{2} + \binom{s-2}{2}$ , because  $m_{d-(s-2)} = m_{\lceil \frac{d}{2} \rceil - 1} = m_{s-3} \geq g^{\lceil \frac{s-3}{2} \rceil + 2}$ .

Note that  $\binom{\lceil \frac{s-3}{2} \rceil + 1}{2} + \binom{s-2}{2} \geq \binom{s-1}{2}$  if  $\binom{\lceil \frac{s-3}{2} \rceil + 1}{2} \geq s-2$ , that is true for every  $s \geq 10$ .  $\square$



**Remark 6.4.** The result of Theorem 6.3 implies  $[0, g^{\lceil \frac{d}{2} \rceil + 2}] \subset G_d$ , for every  $d \geq 18$ . By a direct computation, we check that this fact is true for every  $d \geq 14$ , also when  $m_d < g^{\lceil \frac{d}{2} \rceil + 2}$ .

$d$	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35
$m_d$	43	52	62	73	85	89	102	116	118	133	149	166	184	203	208	228	229	229	250
$g^{\lceil \frac{d}{2} \rceil + 2}$	45	45	55	55	66	66	78	78	91	91	105	105	120	120	136	136	153	153	171

TABLE 1. Values of the sequence  $\{m_d\}_{d \geq 1}$  compared with the values of  $g^{\lceil \frac{d}{2} \rceil + 2}$ , for  $17 \leq d \leq 35$ .

Theorem 6.3 gives a lower bound for the value assumed by  $m_d$ , for every  $d \geq 18$ . Anyway, we can obtain more information by a full application of Lemma 6.1 which, together with the algorithm GENUSSEARCH (see Algorithm 1), leads us to describe an algorithm to compute all the arithmetic genera of the aCM curves of degree  $d$ , avoiding to construct all the finite O-sequences. The strategy consists of the following steps:

- Step 1:** we determine recursively the set of integers  $\tilde{G}_d \subset R_d$  that are certainly aCM genera by Lemma 6.1. Let  $\tilde{G}_1 = \{0\}$ , we have  $\tilde{G}_d = \bigcup_i \tilde{G}_i + \binom{d-i}{2}$ ;
- Step 2:** we determine all the integers of  $R_d$  that are certainly gaps by results in Section 5;
- Step 3:** we use algorithm GENUSSEARCH (Algorithm 1) to investigate the remaining integers.

---

**Algorithm 2** The algorithm for determining the aCM genera of curves with a given degree. A trial version of this algorithm is available at

[www.personalweb.unito.it/paolo.lella/HSC/Finite\\_O-sequences\\_and\\_ACM\\_genus.html](http://www.personalweb.unito.it/paolo.lella/HSC/Finite_O-sequences_and_ACM_genus.html)

---

1: ACMGENERA( $d$ )

**Require:**  $d$ , a positive integer.

**Ensure:** the list of all possible aCM genera of a curve of degree  $d$ .

```

2: genera := {genera determined applying recursively Lemma 6.1};
3: gaps := {gaps determined applying Proposition 5.3 and Proposition 5.10};
4: undecided :=  $\{0, \dots, \binom{d-1}{2}\} \setminus (\text{genera} \cup \text{gaps})$ ;
5: for  $s = 2, \dots, d - 3$  do
6:    $g := \min(\text{undecided})$ ;
7:   while  $g \leq \text{UPPERBOUND}(R_d^s)$  do
8:     if  $g < \text{LOWERBOUND}(R_d^s)$  then
9:       REMOVE( $g$ , undecided);
10:      gaps = gaps  $\cup \{g\}$ ;
11:    else
12:      searching := GENUSSEARCH( $g, \mathcal{T}_d^s$ );
13:      if searching  $\neq \emptyset$  then
14:        REMOVE( $g$ , undecided);
15:        genera = genera  $\cup \{g\}$ ;
16:      end if
17:    end if
18:     $g = \text{NEXT}(g, \text{undecided})$ ;
19:  end while
20: end for
21: return genera;
```

---



$d$	Certain genera	Certain gaps	Undecided values	$ G_d $
25	176 (63.77%)	88 (31.88%)	13 (4.71%)	187 (67.75%)
50	835 (71.00%)	289 (24.57%)	53 (4.51%)	870 (73.98%)
75	2033 (75.27%)	558 (20.66%)	111 (4.11%)	2099 (77.71%)
100	3798 (78.29%)	879 (18.12%)	175 (3.61%)	3894 (80.27%)
125	6129 (80.37%)	1244 (16.31%)	254 (3.33%)	6261 (82.10%)
150	9040 (81.99%)	1653 (14.99%)	334 (3.02%)	9207 (83.50%)
175	12528 (83.24%)	2094 (13.91%)	430 (2.86%)	12734 (84.61%)
200	16610 (84.31%)	2574 (13.07%)	518 (2.63%)	16854 (85.55%)
225	21276 (85.19%)	3084 (12.35%)	617 (2.47%)	21560 (86.32%)
250	26530 (85.92%)	3623 (11.73%)	724 (2.34%)	26856 (86.98%)

TABLE 2. In this table, we report some numerical information about the integers in  $G_d$  up to degree 250. In the first column, there are the number and the percentage of values in  $R_d$  which are aCM genera by an application of Lemma 6.1 (without computing the O-sequences); in the second column, the number and the percentage of gaps determined applying Proposition 5.3 and Proposition 5.10; in the third column, the number and the percentage of values of  $R_d$  for which we have to use the procedure GENUSSEARCH to decide whether they are aCM genera; in the last column, the cardinality of  $G_d$  and its percentage with respect to  $|R_d|$ .

$d$	Step 1	Step 2	Step 3	Algorithm 2	Visit $\mathcal{T}_d$
25	37.336 ms	0.164 ms	38.594 ms	76.094 ms	210.769 ms
50	82.774 ms	0.208 ms	212.868 ms	295.850 ms	15155.87 ms
75	21.734 ms	0.155 ms	458.117 ms	480.006 ms	O.O.M.
100	47.529 ms	0.103 ms	1390.027 ms	1437.659 ms	O.O.M.
125	104.683 ms	0.279 ms	4684.598 ms	4789.56 ms	O.O.M.
150	207.936 ms	0.183 ms	12610.461 ms	12818.58 ms	O.O.M.
175	546.818 ms	0.227 ms	37518.036 ms	38065.081 ms	O.O.M.
200	665.378 ms	0.364 ms	73552.564 ms	74218.306 ms	O.O.M.
225	922.599 ms	0.36 ms	169042.878 ms	169965.837 ms	O.O.M.
250	1395.378 ms	0.179 ms	359836.564 ms	361232.121 ms	O.O.M.

TABLE 3. In this table, we report the results of a test of Algorithm 2 up to degree 250. The first three columns contain the elapsed time (in milliseconds) for **Step 1**, **Step 2** and **Step 3** of Algorithm 2. In the fourth column, there is the total time for the execution (Step 1 + Step 2 + Step 3). The last column contains the time required for determining the set  $G_d$  by performing a complete visit of the tree  $\mathcal{T}_d$  (even for  $d = 75$ , we obtain an Out Of Memory Error). The algorithms are implemented in the Java language and have been run on a MacBook Pro with an Intel Core 2 Duo 2.4 GHz processor.

## 7. CASTELNUOVO-MUMFORD REGULARITY OF CURVES WITH COHEN-MACAULAY POSTULATION

In this section, we show that it is enough to apply the algorithm of search of aCM genera (Algorithm 1) to detect the minimal Castelnuovo-Mumford regularity  $m_{d,g}^{\text{aCM}}$  of a curve with Cohen-Macaulay postulation, given its degree  $d$  and genus  $g$ . A complete answer to the problem of detecting the minimal Castelnuovo-Mumford regularity of a scheme with a given Hilbert polynomial is described in [3].

**Proposition 7.1.**

$$m_{d,g}^{\text{aCM}} = \min \left\{ \rho \mid \begin{array}{l} \rho \text{ is the regularity of an aCM postulation} \\ \text{with Hilbert polynomial } dt + 1 - g \end{array} \right\} + 2$$

*Proof.* Let  $f$  be an aCM function with regularity  $\rho$  and Hilbert polynomial  $dt + 1 - g$ . Then the minimal possible Castelnuovo-Mumford regularity of a curve with Hilbert function  $f$  is  $\rho + 2$ . Indeed, it is enough to observe that by [2, Proposition 2.4] this regularity is strictly greater than  $\rho + 1$  and that, if the curve is aCM, it is exactly  $\rho + 2$ .  $\square$

By Proposition 7.1, the value of  $m_{d,g}^{\text{aCM}}$  is determined by applying Algorithm 1 in order to find an O-sequence  $\mathbf{h}$  of multiplicity  $d$  and  $g(\mathbf{h}) = g$  with the shortest possible length. Notice that if the length of  $\mathbf{h}$  is  $s$ , then the regularity of  $\Sigma^2 \mathbf{h}$  is  $s - 2$ . Thus, we can rewrite the statement in Proposition 7.1 as

$$m_{d,g}^{\text{aCM}} = \min \left\{ s \mid \begin{array}{l} s \text{ is the length of an O-sequence } \mathbf{h} \\ \text{with multiplicity } d \text{ and } g(\mathbf{h}) = g \end{array} \right\}.$$

**Example 7.2.** Let us consider the curves of degree  $d = 15$  and genus  $g = 32$ . There are four O-sequences of multiplicity  $d$  corresponding to aCM curves of genus  $g$ :

$$\begin{aligned} \mathbf{h}_1 &= (1, 4, 3, 2, 1, 1, 1, 1), & \mathbf{h}_3 &= (1, 2, 3, 4, 2, 1, 1, 1), \\ \mathbf{h}_2 &= (1, 3, 3, 2, 2, 2, 1, 1), & \mathbf{h}_4 &= (1, 3, 5, 1, 1, 1, 1, 1). \end{aligned}$$

Hence, the minimal Castelnuovo-Mumford regularity  $m_{d,g}^{\text{aCM}}$  is 8. Applying the results of [3] (see [www.personalweb.unito.it/paolo.lella/HSC/Minimal\\_Hilbert\\_Functions\\_and\\_CM\\_regularity.html](http://www.personalweb.unito.it/paolo.lella/HSC/Minimal_Hilbert_Functions_and_CM_regularity.html)), we notice that the minimal Castelnuovo-Mumford regularity of any projective scheme with Hilbert polynomial  $p(t) = 15t - 31$  is 7.

More generally, in the case of an aCM function  $f$  with regularity  $\rho$  and Hilbert polynomial with odd degree, we have that the minimal possible Castelnuovo-Mumford regularity of a scheme  $X$  with  $H_X = f$  is strictly greater than  $\rho + 1$  (see [2, Proposition 2.4]). If the degree of the Hilbert polynomial is even, an analogous result does not hold, as the following example shows.

**Example 7.3.** The following strongly-stable ideal

$$\begin{aligned} I = (x_6^2, x_5x_6, x_5^2, x_4x_5, x_3x_5, x_2x_5, x_1x_5, x_4^2x_6, x_3x_4x_6, x_2x_4x_6, x_1x_4x_6, x_3^2x_6, x_2x_3x_6, \\ x_1x_3x_6, x_2^3x_6, x_1x_2^2x_6, x_1^2x_2x_6, x_4^4, x_3x_4^3, x_2x_4^3, x_1^4x_6, x_3^3x_4^2, x_3^4x_4, x_3^5) \subset K[x_0, \dots, x_6], \end{aligned}$$

where  $x_0 < x_1 < \dots < x_6$ , defines a non-aCM surface  $X \subset \mathbb{P}^6$  with the aCM function  $H_X = \Sigma^2(1, 4, 3, 1, 1, 1, 1) = (1, 7, 21, 44, \dots, 6t^2 - 10t + 21, \dots)$  of regularity 4 and the Castelnuovo-Mumford regularity of  $X$  is 5.

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